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Note on the instability of a non-uniform vortex sheet

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Howard's semicircle theorem and a variational principle for the instability of unidirectional flow of an inviscid fluid are applied to the non-uniform vortex sheets discussed in a previous paper (Hocking 1964). Certain results of § 5 of that paper are shown to be wrong and the correct results are obtained.

1. The semicircle theorem

Howard (1961) has proved that the complex wave velocity of an unstable disturbance to plane unidirectional flow with velocity $W(y)$ must lie within the semicircle in the upper half-plane which has the range of values of W as diameter. Eckart (1963) has shown that the same result holds for more general flows, including non-planar flows with velocity $W(x, y)$. Because intermediate results are also needed in the following section, a derivation of this result for non-planar flows, slightly different from Eckart's, is given here. The fluid is of uniform density ρ and occupies a cylindrical region with cross-section S in the (x, y)-plane, bounded by the curve C. If the velocity disturbance is $(u, v, w) \exp\{i\alpha(z-ct)\}$ and the pressure disturbance $p \exp\{i\alpha(z-ct)\}$ and if $\mathbf{u}$ is the two-dimensional vector (u, v) and ∇ the two-dimensional gradient operator, the linearized equations of motion and the equation of continuity are

$$i\alpha(W-c)\,\mathbf{u} = -\nabla p/\rho, \tag{1}$$

$$i\alpha(W-c)\,w + \mathbf{u}\,.\,\nabla W = -i\alpha p/\rho, \tag{2}$$

$$\nabla\mathbf{u} + i\alpha w = 0. \tag{3}$$

The equation for p is found from these equations to be

$$\nabla\{(W-c)^{-2}\nabla p\} - \alpha^2(W-c)^{-2}p = 0. \tag{4}$$

The boundary condition that the normal velocity vanishes on C is, because of (1), $\mathbf{n}\,.\,\nabla p = 0$, where $\mathbf{n}$ is normal to C, so that

$$\int(\mathbf{n}\,.\,\nabla p)\,(W-c)^{-2}\tilde{p}\,dC = 0, \tag{5}$$

where $\tilde{p}$ is the complex conjugate of p. Transforming this integral round C to an integral over S, we have

$$\iint\nabla\{(W-c)^{-2}\tilde{p}\nabla p\}\,dS \equiv \iint(W-c)^{-2}\,\nabla\tilde{p}\,.\,\nabla p\,dS + \iint\tilde{p}\nabla\{(W-c)^{-2}\,\nabla p\}\,dS = 0,$$

and, by use of (4), this reduces to

$$\iint (W-c)^{-2}\{\nabla p \,.\, \nabla \tilde{p} + \alpha^2 |p|^2\}\, dS = 0. \tag{6}$$

With $c = c_r + ic_i$ and $c_i > 0$, the real and imaginary parts of this equation give

$$\iint (W-c_r)\,\Phi\, dS = 0, \tag{7}$$

$$\iint \{(W-c_r)^2 - c_i^2\}\,\Phi\, dS = 0, \tag{8}$$

where $$\Phi = |W-c|^{-4}\{|\partial p/\partial x|^2 + |\partial p/\partial y|^2 + \alpha^2 |p|^2\} \geqslant 0.$$

Equations (7) and (8) are exactly the same as those obtained by Howard, apart from the form of Φ. Hence Howard's result, which he derived by integrating the non-negative function $(W-a)(b-W)\Phi$, where a and b are the least and greatest values of W, is also true for non-planar flow, i.e. the complex wave velocity lies within the semicircle which has the range of values of $W(x, y)$ as diameter.

When this result is applied to non-uniform vortex sheets, a second result of a similar kind can be obtained. Suppose that S is divided by a vortex sheet into two regions S_1 and S_2, and that $a_1 \leqslant W \leqslant b_1$ in S_1 and $a_2 \leqslant W \leqslant b_2$ in S_2. If $b_1 < a_2$, so that the ranges of W in the two regions do not overlap, the semicircle theorem shows that c lies within the semicircle with diameter extending from a_1 to b_2. The same argument that led to the semicircle theorem, when applied to the non-positive function $(W-b_1)(a_2-W)\Phi$, shows that c must lie *outside* the semicircle with diameter extending from b_1 to a_2, so there is an inner, as well as an outer, boundary to c. If the ranges of W overlap, i.e. $b_1 > a_2$, the inner boundary disappears.

When the results obtained in a previous paper (Hocking 1964) are tested by these restrictions on the possible values of c, it is found that the values of c obtained in §§ 3, 4 lie in the appropriate regions, but those obtained in § 5 do not. The value of W in that section was

$$W = W_0\{1 + \lambda(2y-\pi)/\pi\} \quad (x > 0),$$
$$= 0 \quad (x < 0),$$

with λ small and the flow confined to the region $0 \leqslant y \leqslant \pi$. The reason for the discrepancy is that a negative sign was omitted from the values of γ_{mn} given in equation (47) and this error, for which the author apologises, has invalidated the subsequent calculation of c and the construction of both figures. When the calculations were repeated with the correct values of γ_{mn}, the value of c with the largest imaginary part was found to be $\frac{1}{2}W_0(1+\lambda)(1+i)$, which lies on the semicircle with the points 0 and $W_0(1+\lambda)$ as the ends of the diameter. The maximum growth rate can be written $\frac{1}{2}\alpha \max(W^i)$, where W^i is the velocity at the interface $x = 0$. In the previous paper, the same result was found to be true generally for large α, and also, for the particular flow considered in § 4, to be true for any α.

2. A variational principle

If $\tilde{p}$ is replaced by p in (5), the analysis which led to (6) gives

$$F(p, W, c) \equiv \iint (W-c)^{-2}\{(\nabla p)^2 + \alpha^2 p^2\}\, dS = 0. \tag{9}$$

If δF_p is the variation in F produced by keeping W and c fixed and replacing p by $p+\delta p$, we have

$$\begin{aligned}\delta F_p &= 2\iint (W-c)^{-2}\{\nabla p \,.\, \nabla\, \delta p + \alpha^2 p\, \delta p\}\, dS \\ &= 2\int (\mathbf{n}\,.\,\nabla p)\,(W-c)^{-2}\, \delta p\, dC - 2\iint \delta p\, [\nabla\{(W-c)^{-2}\nabla p\} - \alpha^2 (w-c)^{-2} p]\, dS,\end{aligned}$$

so that δF_p vanishes for arbitrary δp if p satisfies (4) and the boundary condition. This variational principle can be used to determine the change in the eigenvalues c when a small change is made in W. Suppose p_0 is an eigenfunction corresponding to the eigenvalue c_0 for a flow with velocity W_0, and that $p_0+\delta p$ and $c_0+\delta c$ are the corresponding quantities when $W = W_0+\delta W$. Then

$$F(p_0+\delta p, W_0+\delta W, c_0+\delta c) = F(p_0, W_0, c_0) = 0.$$

Since p_0 satisfies (4), the variational principle shows that $\delta F_p = 0$, and hence $\delta F_W + \delta F_c = 0$, which gives

$$\iint (\delta W - \delta c)\,(W_0 - c_0)^{-3}\{(\nabla p_0)^2 + \alpha^2 p_0^2\}\, dS = 0. \tag{10}$$

In the non-uniform vortex sheets under consideration, the velocity distribution is $W = W_0\{1+\lambda W_1(x,y)\}$, $x > 0$ and $W = 0$, $x < 0$. With λ zero, the value of c is $\frac{1}{2}W_0(1+i)$ and p is an even function of x. Substituting these values in (10), we find that, for λ small, c has the form $\frac{1}{2}W_0(1+i)\,(1+\lambda q)$ with q real and

$$q \iint_{x>0} \{(\nabla p_0)^2 + \alpha^2 p_0^2\}\, dS = \iint_{x>0} W_1\{(\nabla p_0)^2 + \alpha^2 p_0^2\}\, dS, \tag{11}$$

and p_0 satisfies $(\nabla^2 - \alpha^2)\, p_0 = 0$.

The semicircle theorem shows that $\min(W_1) < q < \max(W_1)$, which is also obvious directly from (11). It was shown in § 3 of the previous paper that, for $\alpha \to \infty$, q takes all values between $\min(W_1^i)$ and $\max(W_1^i)$. If the least and greatest values of W_1 are attained at the interface, q can take all values between these values of W_1 and no others. For the flows discussed in §§ 4, 5 of the previous paper, the extreme values of W_1 were ± 1, and these values were attained at the interface. It follows at once that the *complete* range of values of c for both flows is $\frac{1}{2}W_0(1+i)\,(1+\lambda q)$ where q takes all values between -1 and 1. However, the detailed calculations for those two flows showed the additional fact that the upper bound of q is 1 for any given α and not just for $\alpha \to \infty$.

The use of the variational principle was suggested to me by Prof. L. N. Howard and I wish to record my thanks for his helpful suggestions about this problem.

REFERENCES

ECKART, C. 1963 *Phys. Fluids*, **6**, 1042.
HOCKING, L. M. 1964 *J. Fluid Mech.* **18**, 177.
HOWARD, L. N. 1961 *J. Fluid Mech.* **10**, 509.